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A Finite Model Property for Gödel Modal Logics

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Abstract. A new semantics with the finite model property is provided and used to establish decidability for Gödel modal logics based on (crisp or fuzzy) Kripke frames combined locally with Gödel logic. A similar methodology is also used to establish decidability, and indeed co-NP-completeness for a Gödel $\mathsf{S5}$ logic that coincides with the one-variable fragment of first-order Gödel logic.

1 Introduction

Gödel modal logics combine Kripke frames of modal logics with the semantics of the well-known fuzzy (and intermediate) Gödel logic. These logics, in particular, analogues GK (for "fuzzy" frames) and $\mathsf{GK^C}$ (for "crisp" frames) of the modal logic K, have been investigated in some detail by Caicedo and Rodríguez [7, 6] and Metcalfe and Olivetti [13, 14]. More general approaches, focussing mainly on finite-valued modal logics, have been developed by Fitting [9, 10], Priest [15], and Bou et al. [4]. Multimodal variants of GK have also been proposed as the basis for fuzzy description logics in [12] and (restricting to finite models) [3].

Axiomatizations were obtained for the box and diamond fragments of GK (where the box fragments of GK and $\mathsf{GK^C}$ coincide) in [7] and for the diamond fragment of $\mathsf{GK^C}$ in [14]. It was subsequently shown in [6] that the full logic GK is axiomatized either by adding the Fischer Servi axioms for intuitionistic modal logic IK (see [8]) to the union of the axioms for both fragments, or by adding the prelinearity axiom for Gödel logic to IK. Decidability of the diamond fragment of GK was established in [7], using the fact that the fragment has the finite model property with respect to its Kripke semantics. This finite model property fails for the box fragment of GK and $\mathsf{GK^C}$ and the diamond fragment of $\mathsf{GK^C}$, but decidability and PSPACE-completeness for these fragments was established in [13, 14] using analytic Gentzen-style proof systems.

The first main contribution of this paper is to establish the decidability of validity in full GK and $\mathsf{GK^C}$ by providing an alternative Kripke semantics for

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these logics that have the same valid formulas as the original semantics, but also admit the finite model property. The key idea of the new semantics is to restrict evaluations of modal formulas at a given world to a particular finite set of truth values. We then use a similar strategy to establish decidability, and indeed co-NP completeness, for the crisp Gödel modal logic $\mathsf{GS5^C}$ based on $\mathsf{S5}$ frames where accessibility is an equivalence relation. Moreover, this logic, an extension of the intuitionistic modal logic MIPC of Bull [5] and Prior [16] with prelinearity and a further modal axiom, corresponds exactly to the one-variable fragment of first-order Gödel logic (see [11]).

2 Gödel Modal Logics

Gödel modal logics are defined based on a language $\mathcal{L}_{\Box\Diamond}$ consisting of a fixed countably infinite set Var of (propositional) variables, denoted $p, q, \ldots$, binary connectives $\to$, $\land$, $\lor$, constants $\bot$, $\top$, and unary operators $\Box$ and $\Diamond$. The set of *formulas* $\mathrm{Fml}_{\Box\Diamond}$, with arbitrary members denoted $\varphi, \psi, \chi, \ldots$ is defined inductively as usual, as are *subformulas* of formulas. We call formulas of the form $\Box\varphi$ and $\Diamond\varphi$ *box-formulas* and *diamond-formulas*, respectively, and fix the *length* of a formula φ, denoted $\ell(\varphi)$, to be the number of symbols occurring in φ. We also define $\neg\varphi = \varphi \to \bot$ and let $\mathrm{Var}(\varphi)$ denote the set of all variables occurring in the formula φ.

The standard semantics of Gödel logic is characterized by the Gödel t-norm min and its residuum $\to_{\mathsf{G}}$, defined on the real unit interval $[0,1]$ by

$$x \to_{\mathsf{G}} y = \begin{cases} y & \text{if } x > y \\ 1 & \text{otherwise.} \end{cases}$$

The Gödel modal logics GK and $\mathsf{GK^C}$ are defined semantically as generalizations of the modal logic K where connectives behave at a given world as in Gödel logic.

A *fuzzy Kripke frame* is a pair $\mathfrak{F} = \langle W, R\rangle$ where W is a non-empty set of *worlds* and $R\colon W \times W \to [0,1]$ is a binary *fuzzy accessibility relation* on W. If $Rxy \in \{0,1\}$ for all $x, y \in W$, then R is called *crisp* and $\mathfrak{F}$, a *crisp Kripke frame*. In this case, we often write $R \subseteq W \times W$ and Rxy to mean $Rxy = 1$.

A GK-*model* is a triple $\mathfrak{M} = \langle W, R, V\rangle$, where $\langle W, R\rangle$ is a fuzzy Kripke frame and $V\colon \mathrm{Var} \times \mathrm{W} \to [0,1]$ is a mapping, called a *valuation*, extended to $V\colon \mathrm{Fml}_{\Box\Diamond} \times \mathrm{W} \to [0,1]$ as follows:

$$\begin{aligned}
V(\bot, x) &= 0 \\
V(\top, x) &= 1 \\
V(\varphi \to \psi, x) &= V(\varphi, x) \to_{\mathsf{G}} V(\psi, x) \\
V(\varphi \land \psi, x) &= \min(V(\varphi, x), V(\psi, x)) \\
V(\varphi \lor \psi, x) &= \max(V(\varphi, x), V(\psi, x)) \\
V(\Box\varphi, x) &= \inf\{Rxy \to_{\mathsf{G}} V(\varphi, y) : y \in W\} \\
V(\Diamond\varphi, x) &= \sup\{\min(Rxy, V(\varphi, y)) : y \in W\}.
\end{aligned}$$

A $\mathsf{GK^C}$-*model* satisfies the extra condition that $\langle W, R\rangle$ is a crisp Kripke frame. In this case, the conditions for $\Box$ and $\Diamond$ may also be read as

$$V(\Box\varphi, x) = \inf(\{1\} \cup \{V(\varphi, y) : Rxy\})$$
$$V(\Diamond\varphi, x) = \sup(\{0\} \cup \{V(\varphi, y) : Rxy\}).$$

A formula $\varphi \in \mathrm{Fml}_{\Box\Diamond}$ is *valid* in a GK-model $\mathfrak{M} = \langle W, R, V\rangle$ if $V(\varphi, x) = 1$ for all $x \in W$. If φ is valid in all L-models for some logic L (in particular GK or $\mathsf{GK^C}$), then φ is said to be L-*valid*, written $\models_{\mathsf{L}} \varphi$.

It is shown in [7] that validity in the box and diamond fragments of GK are axiomatized by extending any axiom system for Gödel logic (e.g., intuitionistic logic plus the prelinearity axiom $(\varphi \to \psi) \lor (\psi \to \varphi)$) with, respectively:

$$\begin{array}{lcl} \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) & \quad\text{and}\quad & \Diamond(\varphi \lor \psi) \to (\Diamond\varphi \lor \Diamond\psi) \\ \neg\neg\Box\varphi \to \Box\neg\neg\varphi & & \Diamond\neg\neg\varphi \to \neg\neg\Diamond\psi \\ \varphi \;/\; \Box\varphi & & \neg\Diamond\bot \\ & & \varphi \to \psi \;/\; \Diamond\varphi \to \Diamond\psi. \end{array}$$

Moreover, it was shown in [6] that extending the union of these axiomatizations with the following Fischer Servi axioms (see [8]) axiomatizes the full logic GK (equivalently, extending the intuitionistic modal logic IK with prelinearity):

$$\Diamond(\varphi \to \psi) \to (\Box\varphi \to \Diamond\psi)$$
$$(\Diamond\varphi \to \Box\psi) \to \Box(\varphi \to \psi).$$

The box fragment of $\mathsf{GK^C}$ coincides with the box fragment of GK [7], while the diamond fragment of $\mathsf{GK^C}$ is axiomatized by adding the rule $\chi \lor (\varphi \to \psi) \;/\; \Diamond\chi \lor (\Diamond\varphi \to \Diamond\psi)$ to the diamond fragment of GK [14]. No axiomatization has yet been found for the full logic $\mathsf{GK^C}$.

Let us agree to call a model *finite* if its set of worlds is finite, and say that a logic has the *finite model property* if validity in the logic coincides with validity in all finite models of the logic. In [7], it is shown that the formula $\Box\neg\neg p \to \neg\neg\Box p$ is valid in all finite GK-models, but not in the infinite crisp model $\langle \mathbb{N}, R, V\rangle$ where $Rxy = 1$ for all $x, y \in \mathbb{N}$ and $V(p, x) = 1/(x+1)$ for all $x \in \mathbb{N}$. That is, neither GK nor $\mathsf{GK^C}$ has the finite model property. The diamond fragment of GK (but not of $\mathsf{GK^C}$) does have the finite model property and this can be used to show that validity in the fragment is decidable [7]. Decidability and indeed PSPACE-completeness of validity in the box and diamond fragments of both GK and $\mathsf{GK^C}$ was established in [13, 14] using analytic Gentzen-style proof systems. However, decidability of validity in the full logics GK and $\mathsf{GK^C}$ has not as yet been established, and indeed will be the main goal of the following two sections.

3 A New Semantics and Finite Model Property

In order for a $\mathsf{GK^C}$-model to render the formula $\varphi = \Box\neg\neg p \to \neg\neg\Box p$ invalid at a world x, there must be values of p at worlds accessible to x that form an infinite descending sequence tending to but never reaching 0. This ensures that the infinite model falsifies φ, but also that no particular world acts as a "witness" to the value of $\Box p$. Our strategy in what follows will be to redefine models to allow only a finite number of values at each world that can be taken by box-formulas and diamond-formulas. A formula such as $\Box p$ can then be "witnessed" at a world where the value of p is merely "sufficiently close" to the value of $\Box p$.

Let us define a GFK-*model* as a quadruple $\mathfrak{M} = \langle W, R, T, V\rangle$, where $\langle W, R, V\rangle$ is a GK-model and $T\colon W \to \mathcal{P}_{<\omega}([0,1])$ is a function from worlds to finite sets of truth values satisfying $\{0,1\} \subseteq T(x) \subseteq [0,1]$ for all $x \in W$. If $\langle W, R, V\rangle$ is also a $\mathsf{GK^C}$-model, then $\mathfrak{M}$ will be called a $\mathsf{GFK^C}$-*model*.

The GFK-valuation V is extended to formulas using the same clauses for non-modal connectives as for GK-valuations, together with the revised modal connective clauses:

$$V(\Box\varphi, x) = \max\{r \in T(x) : r \leq \inf\{Rxy \to_{\mathsf{G}} V(\varphi, y) : y \in W\}\}$$
$$V(\Diamond\varphi, x) = \min\{r \in T(x) : r \geq \sup\{\min(Rxy, V(\varphi, y)) : y \in W\}\}.$$

As before, a formula $\varphi \in \mathrm{Fml}_{\Box\Diamond}$ is *valid* in a GFK-model $\mathfrak{M} = \langle W, R, T, V\rangle$ if $V(\varphi, x) = 1$ for all $x \in W$, written $\mathfrak{M} \models_{\mathsf{GFK}} \varphi$.

Observe now that for the formula $\varphi = \Box\neg\neg p \to \neg\neg\Box p$, there are very simple finite $\mathsf{GFK^C}$-counter-models: for example, $\mathfrak{M}_0 = \langle W, R, T, V\rangle$ with $W = \{a\}$, $Raa = 1$, $T(a) = \{0,1\}$, and $V(p,a) = \frac{1}{2}$. It is easy to see that $V(\neg p, a) = 0$, $Raa \to_{\mathsf{G}} V(\neg\neg p, a) = 1$, and so $V(\Box\neg\neg p, a) = 1$. Moreover, $V(\Box p, a) = 0$ (since $Raa \to_{\mathsf{G}} V(p,a) = \frac{1}{2}$, and 0 is the next smaller element of $T(a)$); hence $V(\neg\Box p, a) = 1$ and $V(\neg\neg\Box p, a) = 0$. So $1 = V(\Box\neg\neg p, a) > V(\neg\neg\Box p, a) = 0$ and $\mathfrak{M}_0 \not\models_{\mathsf{GFK^C}} \Box\neg\neg p \to \neg\neg\Box p$.

Of course, such an observation is useful only if the new semantics characterizes the same logics. For convenience, let us agree to write $W_{\mathfrak{M}}$, $R_{\mathfrak{M}}$, $T_{\mathfrak{M}}$, and $V_{\mathfrak{M}}$ for, respectively, the set of worlds, accessibility relation, truth value function, and valuation function of an L-model $\mathfrak{M}$ where $\mathsf{L} \in \{\mathsf{GK}, \mathsf{GK^C}, \mathsf{GFK}, \mathsf{GFK^C}\}$. In the next section, we prove the following:

Theorem 1. *For each $\varphi \in \mathrm{Fml}_{\Box\Diamond}$:*

(a) $\models_{\mathsf{GK}} \varphi$ *iff* $\models_{\mathsf{GFK}} \varphi$ *iff φ is valid in all GFK-models $\mathfrak{M}$ satisfying $|W_{\mathfrak{M}}| \leq (\ell(\varphi)+2)^{\ell(\varphi)}$ and $|T_{\mathfrak{M}}(x)| \leq \ell(\varphi)+2$ for all $x \in W_{\mathfrak{M}}$.*

(b) $\models_{\mathsf{GK^C}} \varphi$ *iff* $\models_{\mathsf{GFK^C}} \varphi$ *iff φ is valid in all $\mathsf{GFK^C}$-models $\mathfrak{M}$ satisfying $|W_{\mathfrak{M}}| \leq (\ell(\varphi)+2)^{\ell(\varphi)}$ and $|T_{\mathfrak{M}}(x)| \leq \ell(\varphi)+2$ for all $x \in W_{\mathfrak{M}}$.*

For decidability, we then reason as follows. Observe first that for any finite GFK-model $\mathfrak{M}$ and formula φ, the values taken by the subformulas of φ and the fuzzy accessibility relation $R_{\mathfrak{M}}$ are contained in the finite set

$$U = \bigcup_{x \in W_{\mathfrak{M}}} (\{V_{\mathfrak{M}}(p,x) : p \in \mathrm{Var}(\varphi)\} \cup \{R_{\mathfrak{M}}xy : y \in W_{\mathfrak{M}}\} \cup T_{\mathfrak{M}}(x)).$$

Moreover, using Lemma 1(c) below, we may assume without loss of generality that $U = \{0, \frac{1}{|U|-1}, \ldots, \frac{|U|-2}{|U|-1}, 1\}$. Hence, by Theorem 1, to check whether φ is

GK-valid or $\mathsf{GK^C}$-valid, it suffices to consider finitely many different finite GFK-models or $\mathsf{GFK^C}$-models $\mathfrak{M}$ (with $|W_{\mathfrak{M}}| \leq (\ell(\varphi)+2)^{\ell(\varphi)}$). So we obtain:

Theorem 2. *Validity in* GK *and* $\mathsf{GK^C}$ *are decidable.*

4 Proof of Theorem 1

We begin by fixing some useful notation. For a fuzzy Kripke frame $\langle W, R\rangle$, we define a crisp relation $R^+ = \{(x,y) \in W^2 : Rxy > 0\}$ and let $R^+[x] = \{y \in W : R^+xy\}$ for each $x \in W$.

We call $\langle W', R'\rangle$ a *subframe* of $\langle W, R\rangle$, written $\langle W', R'\rangle \subseteq \langle W, R\rangle$, if $W' \subseteq W$ and R' is R restricted to W'. A *submodel* $\widehat{\mathfrak{M}}$ of a model $\mathfrak{M}$ is based on a subframe $\langle W_{\widehat{\mathfrak{M}}}, R_{\widehat{\mathfrak{M}}}\rangle \subseteq \langle W_{\mathfrak{M}}, R_{\mathfrak{M}}\rangle$, with $T_{\widehat{\mathfrak{M}}}$ (if appropriate) and $V_{\widehat{\mathfrak{M}}}$ being, respectively, $T_{\mathfrak{M}}$ and $V_{\mathfrak{M}}$ restricted to $W_{\widehat{\mathfrak{M}}}$. In particular, given $X \subseteq W_{\mathfrak{M}}$, the *submodel of* $\mathfrak{M}$ *generated by* X is the smallest submodel $\widehat{\mathfrak{M}}$ of $\mathfrak{M}$ satisfying $X \subseteq W_{\widehat{\mathfrak{M}}}$ and for all $x \in W_{\widehat{\mathfrak{M}}}$, if $y \in R^+_{\mathfrak{M}}[x]$ then $y \in W_{\widehat{\mathfrak{M}}}$. Also, $\mathfrak{M}$ will be called a *tree-model* if $\langle W_{\mathfrak{M}}, R^+_{\mathfrak{M}}\rangle$ is a tree, and the *height* $hg(\mathfrak{M})$ of $\mathfrak{M}$ is the height of $\langle W_{\mathfrak{M}}, R^+_{\mathfrak{M}}\rangle$ (possibly ∞).

Parts (a) and (b) of the following lemma generalize well-known results for the modal logic K (see, e.g., [2]), while part (c) generalizes a useful result from [14] (Lemma 3.1). Their proofs will be omitted here, but follow very closely the ideas of the previous proofs from the references.

Lemma 1. *Let* $\mathsf{L} \in \{\mathsf{GK}, \mathsf{GK^C}, \mathsf{GFK}, \mathsf{GFK^C}\}$ *and let* $\mathfrak{M}$ *be an* L*-model.*

(a) *Given any generated submodel* $\widehat{\mathfrak{M}}$ *of* $\mathfrak{M}$*,* $V_{\widehat{\mathfrak{M}}}(\varphi, x) = V_{\mathfrak{M}}(\varphi, x)$ *for all* $x \in W_{\widehat{\mathfrak{M}}}$*, and* $\varphi \in \mathrm{Fml}_{\Box\Diamond}$*.*

(b) *Given* $x_0 \in W_{\mathfrak{M}}$ *and* $\varphi \in \mathrm{Fml}_{\Box\Diamond}$*, there is an* L*-tree-model* $\widehat{\mathfrak{M}}$ *with root* $\widehat{x}_0$ *and* $hg(\widehat{\mathfrak{M}}) \leq \ell(\varphi)$ *satisfying* $V_{\widehat{\mathfrak{M}}}(\varphi, \widehat{x}_0) = V_{\mathfrak{M}}(\varphi, x_0)$*.*

(c) *Given an order-embedding* $h\colon [0,1] \to [0,1]$ *satisfying* $h(0) = 0$ *and* $h(1) = 1$*, consider* $\widehat{\mathfrak{M}}$ *with* $W_{\widehat{\mathfrak{M}}} = W_{\mathfrak{M}}$*,* $R_{\widehat{\mathfrak{M}}}xy = h(R_{\mathfrak{M}}xy)$*,* $T_{\widehat{\mathfrak{M}}}(x) = h(T_{\mathfrak{M}}(x))$*, and* $V_{\widehat{\mathfrak{M}}}(p, x) = h(V_{\mathfrak{M}}(p, x))$ *for all* $x, y \in W_{\mathfrak{M}}$ *and* $p \in \mathrm{Var}$*. Then* $V_{\widehat{\mathfrak{M}}}(\varphi, x) = h(V_{\mathfrak{M}}(\varphi, x))$ *for all* $\varphi \in \mathrm{Fml}_{\Box\Diamond}$ *and* $x \in W_{\mathfrak{M}}$*.*

Note that the tree in (b), although it is of finite height, can still be infinitely branching and thus contain infinitely many nodes (i.e., worlds).

We now provide the key construction of a GK-tree-model taking the same values for formulas at its root as a given GFK-tree-model. Note first that the original GFK-model without the function T cannot play this role in general since the infimum or supremum required for calculating the value of a box or diamond formula might not be in the set $T(x_0)$ (where x_0 is the root world). This problem is resolved by taking infinitely many order-isomorphic copies of the original GFK-model (without T) in such a way that the open intervals between members of $T(x_0)$ are "squeezed" closer to either their lower or upper bounds. The obtained infima and suprema will then coincide with the next smaller or larger member of $T(x_0)$, that is, the required values of the formulas at x_0 in the original GFK-model.

Lemma 2. *For any* GFK*-tree-model* $\mathfrak{M} = \langle W, R, T, V\rangle$ *of finite height with root* x_0*, there is a* GK*-tree-model* $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V}\rangle$ *with root* $\widehat{x}_0$*, such that* $\widehat{V}(\varphi, \widehat{x}_0) = V(\varphi, x_0)$ *for all* $\varphi \in \mathrm{Fml}_{\Box\Diamond}$*. Moreover, if* $\mathfrak{M}$ *is crisp, then so is* $\widehat{\mathfrak{M}}$*.*

Proof. The lemma is proved by induction on $hg(\mathfrak{M})$. The base case $hg(\mathfrak{M}) = 0$ is immediate, fixing $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V}\rangle$ with $\widehat{W} = W = \{x_0\}$, $\widehat{R} = R = \emptyset$, and $\widehat{V} = V$. For the inductive step $hg(\mathfrak{M}) = n + 1$, define for all $y \in R^+[x_0]$, $\mathfrak{M}_y = \langle W_y, R_y, T_y, V_y\rangle$ as the submodel of $\mathfrak{M}$ generated by $\{y\}$. That is, $\mathfrak{M}_y$ is a GFK-tree-model of finite height with root y, $hg(\mathfrak{M}_y) \leq n$, and, by Lemma 1(a), $V_y(\varphi, x) = V(\varphi, x)$ for all $x \in W_y$ and $\varphi \in \mathrm{Fml}_{\Box\Diamond}$. So, by the induction hypothesis, for each $y \in R^+[x_0]$, there is a GK-tree-model $\widehat{\mathfrak{M}}_y = \langle \widehat{W}_y, \widehat{R}_y, \widehat{V}_y\rangle$ (crisp if $\mathfrak{M}$ is crisp) with root $\widehat{y}$ such that $\widehat{V}_y(\varphi, \widehat{y}) = V_y(\varphi, y)$ $(= V(\varphi, y))$ for all $\varphi \in \mathrm{Fml}_{\Box\Diamond}$.

We now define infinitely many copies of our models $\widehat{\mathfrak{M}}_y$ such that at each copy, all the values of our formulas (and fuzzy accessibility relation) get "squeezed" closer and closer towards the next smaller (or next larger) element of $T(x_0)$. This is achieved by defining for each $k \in \mathbb{Z}^+$, an order-embedding (using Lemma 1(c)) that "squeezes" the open interval between two members α_i and α_{i+1} of $T(x_0)$ into the interval $(\alpha_i, \alpha_i + \frac{1}{k})$ (or $(\alpha_{i+1} - \frac{1}{k}, \alpha_{i+1})$), which gets infinitely small as k approaches infinity.

More formally, consider $T(x_0) = \{\alpha_1, \ldots, \alpha_m\}$ with $0 = \alpha_1 < \ldots < \alpha_m = 1$ and define a family of order-embeddings $\{h_k\}_{k\in\mathbb{Z}^+}$ from $[0, 1]$ into $[0, 1]$ satisfying $h_k(0) = 0$ and $h_k(1) = 1$, such that

$$
\begin{array}{rll}
h_k(\alpha_i) = \alpha_i & & \text{for all } i \leq m \text{ and } k \in \mathbb{Z}^+ \\
h_k[(\alpha_i, \alpha_{i+1})] = (\alpha_i, \min(\alpha_i + \frac{1}{k}, \alpha_{i+1})) & & \text{for all } i \leq m-1 \text{ and even } k \in \mathbb{Z}^+ \\
h_k[(\alpha_i, \alpha_{i+1})] = (\max(\alpha_i, \alpha_{i+1} - \frac{1}{k}), \alpha_{i+1}) & & \text{for all } i \leq m-1 \text{ and odd } k \in \mathbb{Z}^+.
\end{array}
$$

Furthermore, for each $y \in R^+[x_0]$ and $k \in \mathbb{Z}^+$, we define a GK-model $\widehat{\mathfrak{M}}^k_y = \langle \widehat{W}^k_y, \widehat{R}^k_y, \widehat{V}^k_y\rangle$ such that for each $k \in \mathbb{Z}^+$ and $y \in R^+[x_0]$:

(1) $\widehat{W}^k_y$ is a copy of $\widehat{W}_y$ with distinct worlds, where $\widehat{x}^k_y$ is the corresponding copy of $\widehat{x}_y$ (the root is denoted by $\widehat{y}^k$)

(2) $\widehat{R}^k_y \widehat{x}^k_y \widehat{z}^k_y = h_k(\widehat{R}_y \widehat{x}_y \widehat{z}_y)$, for all $\widehat{x}^k_y, \widehat{z}^k_y \in \widehat{W}^k_y$

(3) $\widehat{V}^k_y(\varphi, \widehat{x}^k_y) = h_k(\widehat{V}_y(\varphi, \widehat{x}_y))$ for all $\varphi \in \mathrm{Fml}_{\Box\Diamond}$ and $\widehat{x}^k_y \in \widehat{W}^k_y$.

Note that for all $y \in R^+[x_0]$, $\widehat{x}_y, \widehat{z}_y \in \widehat{W}_y$, and $\varphi \in \mathrm{Fml}_{\Box\Diamond}$, if $\widehat{R}_y \widehat{x}_y \widehat{z}_y \to_{\mathsf{G}} \widehat{V}_y(\varphi, \widehat{x}_y) \in (\alpha_i, \alpha_{i+1})$, then $\widehat{R}^k_y \widehat{x}^k_y \widehat{z}^k_y \to_{\mathsf{G}} \widehat{V}^k_y(\varphi, \widehat{x}^k_y) \in (\alpha_i, \alpha_i + \frac{1}{k})$, for each even $k \in \mathbb{Z}^+$, and if $\min(\widehat{R}_y \widehat{x}_y \widehat{z}_y, \widehat{V}_y(\varphi, \widehat{x}_y)) \in (\alpha_i, \alpha_{i+1})$, then $\min(\widehat{R}^k_y \widehat{x}^k_y \widehat{z}^k_y, \widehat{V}^k_y(\varphi, \widehat{x}^k_y)) \in (\alpha_{i+1} - \frac{1}{k}, \alpha_{i+1})$, for each odd $k \in \mathbb{Z}^+$.

We now define the GK-tree-model $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V}\rangle$ with

$$\widehat{W} = \bigcup_{y\in R^+[x_0]} \bigcup_{k\in\mathbb{Z}^+} \widehat{W}_y^k \cup \{\widehat{x}_0\}$$

$$\widehat{R}xz = \begin{cases} h_k(Rx_0y) & \text{if } x = \widehat{x}_0 \text{ and } z = \widehat{y}^k \text{ for some } y \in R^+[x_0] \text{ and } k \in \mathbb{Z}^+ \\ \widehat{R}_y^k xz & \text{if } x, z \in \widehat{W}_y^k \text{ for some } y \in R^+[x_0] \text{ and } k \in \mathbb{Z}^+ \\ 0 & \text{otherwise} \end{cases}$$

$$\widehat{V}(p,x) = \begin{cases} V(p,x_0) & \text{if } x = \widehat{x}_0 \\ \widehat{V}_y^k(p,x) & \text{if } x \in \widehat{W}_y^k \text{ for some } y \in R^+[x_0] \text{ and } k \in \mathbb{Z}^+. \end{cases}$$

If $\mathfrak{M}$ is crisp, then for all $y \in R^+[x_0]$, $\widehat{\mathfrak{M}}_y$ is crisp and so also are $\widehat{\mathfrak{M}}_y^k$ for all $k \in \mathbb{Z}^+$. Hence, by construction, $\widehat{\mathfrak{M}}$ is crisp. Moreover, $\widehat{V}_y^k(\varphi, \widehat{x}_y^k) = \widehat{V}(\varphi, \widehat{x}_y^k)$ for all $\varphi \in \mathrm{Fml}_{\Box\Diamond}$ and $\widehat{x}_y^k \in \widehat{W} \setminus \{\widehat{x}_0\}$.

Now we prove that $\widehat{V}(\varphi, \widehat{x}_0) = V(\varphi, x_0)$ for all $\varphi \in \mathrm{Fml}_{\Box\Diamond}$, proceeding by induction on $\ell(\varphi)$. The base case $\ell(\varphi) = 1$ follows directly from the definition of $\widehat{V}$. For the inductive step, the cases for the non-modal connectives follow easily using the induction hypothesis. Let us just consider the case $\varphi = \Box\psi$, the case $\varphi = \Diamond\psi$ being very similar. There are two possibilities. Suppose first that

$$V(\Box\psi, x_0) = \max\{r \in T(x_0) : r \leq \inf\{Rx_0y \to_{\mathsf{G}} V(\psi, y) : y \in W\}\} = 1.$$

Then for all $y \in R^+[x_0]$, $Rx_0y \leq V(\psi, y)$ and by Lemma 1(a), $V(\psi, y) = V_y(\psi, y) = \widehat{V}_y(\psi, \widehat{y})$. Thus $Rx_0y \leq \widehat{V}_y(\psi, \widehat{y})$ and therefore, for all $k \in \mathbb{Z}^+$ and $y \in R^+[x_0]$,

$$\widehat{R}\widehat{x}_0\widehat{y}^k = h^k(Rx_0y) \leq h^k(\widehat{V}_y(\psi, \widehat{y})) = \widehat{V}_y^k(\psi, \widehat{y}^k) = \widehat{V}(\psi, \widehat{y}^k).$$

It follows that

$$\widehat{V}(\Box\psi, \widehat{x_0}) = \inf\{\widehat{R}\widehat{x}_0z \to_{\mathsf{G}} \widehat{V}(\psi, z) : z \in \widehat{W}\} = 1 = V(\Box\psi, x_0).$$

Now suppose that $V(\Box\psi, x_0) = \alpha_i < 1$ for some $i \leq m-1$. Then $Rx_0z \to_{\mathsf{G}} V(\psi, z) \geq \alpha_i$ for all $z \in W$, and thus, $(\star)$, $\widehat{R}\widehat{x}_0z \to_{\mathsf{G}} \widehat{V}(\psi, z) \geq \alpha_i$ for all $z \in \widehat{W}$, by construction using the order-embeddings $\{h_k\}_{k\in\mathbb{Z}^+}$.

There are two subcases. First, suppose that there is at least one $y \in W$ such that $Rx_0y \to_{\mathsf{G}} V(\psi, y) = \alpha_i$; call it y_0. This means that $Rx_0y_0 > V(\psi, y_0) = \alpha_i$ and for all $k \in \mathbb{Z}^+$, $\widehat{V}(\psi, \widehat{y}_0^k) = \widehat{V}_{y_0}^k(\psi, \widehat{y}_0^k) = h_k(\widehat{V}_{y_0}(\psi, \widehat{y}_0)) = h_k(V_{y_0}(\psi, y_0)) = h_k(V(\psi, y_0)) = h_k(\alpha_i) = \alpha_i$. Since $Rx_0y > \alpha_i$, also for all $k \in \mathbb{Z}^+$, $\widehat{R}\widehat{x}_0\widehat{y}_0^k = h_k(Rx_0y_0) > \alpha_i = \widehat{V}(\psi, \widehat{y}_0^k)$, and hence, using $(\star)$,

$$\widehat{V}(\Box\psi, \widehat{x}_0) = \inf\{\widehat{R}\widehat{x}_0z \to_{\mathsf{G}} \widehat{V}(\psi, z) : z \in \widehat{W}\} = \alpha_i = V(\Box\psi, x_0).$$

Now suppose that $Rx_0y \to_{\mathsf{G}} V(\psi, y) > \alpha_i$ for all $y \in W$. Since $V(\Box\psi, x_0) = \max\{r \in T(x_0) : r \leq \inf\{Rx_0y \to_{\mathsf{G}} V(\psi, y) : y \in W\}\} = \alpha_i$, there is at least one $y \in W$ such that $Rx_0y \to_{\mathsf{G}} V(\psi, y) \in (\alpha_i, \alpha_{i+1})$; call it y_0. Then, by construction, for any $\varepsilon > 0$ there is a $k \in \mathbb{Z}^+$ such that $h_k(Rx_0y_0 \to_{\mathsf{G}} V(\psi, y_0)) = h_k(Rx_0y_0) \to_{\mathsf{G}} h_k(V(\psi, y_0)) = \widehat{R}\widehat{x}_0\widehat{y}_0^k \to_{\mathsf{G}} \widehat{V}(\psi, \widehat{y}_0^k) \in (\alpha_i, \alpha_i + \varepsilon)$. Using $(\star)$,

$$\widehat{V}(\Box\psi, \widehat{x}_0) = \inf\{\widehat{R}\widehat{x}_0z \to_{\mathsf{G}} \widehat{V}(\psi, z) : z \in \widehat{W}\} = \alpha_i = V(\Box\psi, x_0). \qquad \Box$$

A subset $\Sigma \subseteq \mathrm{Fml}_{\Box\Diamond}$ will be called a *fragment* iff it is closed with respect to taking subformulas and contains $\bot$ and $\top$. For a formula $\varphi \in \mathrm{Fml}_{\Box\Diamond}$, we let $\Sigma(\varphi)$ be the smallest fragment containing φ. Clearly, $|\Sigma(\varphi)| \leq \ell(\varphi) + 2$.

We now show that given any finite fragment Σ and GK-tree-model $\mathfrak{M}$, we are able to "prune" (i.e., remove branches from) $\mathfrak{M}$ and introduce a suitable function T to obtain a finite GFK-tree-model $\widehat{\mathfrak{M}}$ such that the evaluations of formulas in Σ at the roots of $\mathfrak{M}$ and $\widehat{\mathfrak{M}}$ coincide.

Lemma 3. *Let $\Sigma \subseteq \mathrm{Fml}_{\Box\Diamond}$ be a finite fragment. Then for any GK-tree-model $\mathfrak{M} = \langle W, R, V\rangle$ of finite height with root x_0, there is a finite GFK-tree-model $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{T}, \widehat{V}\rangle$ with $\langle \widehat{W}, \widehat{R}\rangle \subseteq \langle W, R\rangle$, root $x_0 \in \widehat{W}$, $|\widehat{W}| \leq |\Sigma|^{hg(\mathfrak{M})}$, and $|\widehat{T}(x)| \leq |\Sigma|$ for all $x \in \widehat{W}$, such that $\widehat{V}(\varphi, x_0) = V(\varphi, x_0)$ for all $\varphi \in \Sigma$. Moreover, if $\mathfrak{M}$ is crisp, then so is $\widehat{\mathfrak{M}}$.*

Proof. Let $\Sigma_\Box$ be the set of all box-formulas in Σ, $\Sigma_\Diamond$ the set of all diamond-formulas in Σ, and Σ_{Var} the set of all variables in Σ. Let us also define $V_x[\Delta] = \{V(\varphi, x) : \varphi \in \Delta\}$ for any $x \in W$ and $\Delta \subseteq \mathrm{Fml}_{\Box\Diamond}$. We prove the lemma by induction on $hg(\mathfrak{M})$. For the base case, it suffices to define $\widehat{W} = W$, $\widehat{R} = R = \emptyset$, $\widehat{V} = V$, and $\widehat{T}(x_0) = \{0, 1\}$.

For the induction step $hg(\mathfrak{M}) = n + 1$, consider for each $y \in R^+[x_0]$, the submodel $\mathfrak{M}_y = \langle W_y, R_y, V_y\rangle$ of $\mathfrak{M}$ generated by $\{y\}$. It is clear that each $\mathfrak{M}_y$ is a GK-tree-model of finite height with root y and $hg(\mathfrak{M}_y) \leq n$. Hence, by the induction hypothesis, for each $y \in R^+[x_0]$ there is a finite GFK-tree model $\widehat{\mathfrak{M}}_y = \langle \widehat{W}_y, \widehat{R}_y, \widehat{T}_y, \widehat{V}_y\rangle$ with $\langle \widehat{W}_y, \widehat{R}_y\rangle \subseteq \langle W_y, R_y\rangle$ and root y, such that for all $\varphi \in \Sigma$: $\widehat{V}_y(\varphi, y) = V_y(\varphi, y)$ $(= V(\varphi, y))$. Moreover, we know for all $y \in R^+[x_0]$ that $|\widehat{W}_y| \leq |\Sigma|^n$ and $|\widehat{T}_y(x)| \leq |\Sigma|$ for all $x \in \widehat{W}_y$.

We now choose a finite number of appropriate $y \in R^+[x_0]$ in order to build our finite GFK-model. To this end, note that we can view $V_{x_0}[\Sigma_\Box \cup \Sigma_\Diamond] \cup \{0, 1\}$ as a finite set $\{\alpha_1, \ldots, \alpha_m\}$ with $0 = \alpha_1 < \ldots < \alpha_m = 1$. Then, for each $\Box\psi \in \Sigma_\Box$, such that $V(\Box\psi, x_0) = \alpha_i < 1$, choose a $y = y_{\Box\psi} \in R^+[x_0]$ such that $Rx_0y_{\Box\psi} \to_{\mathsf{G}} V(\psi, y_{\Box\psi}) < \alpha_{i+1}$, and for each $\Diamond\psi \in \Sigma_\Diamond$, such that $V(\Diamond\psi, x) = \alpha_i > 0$, choose a $y = y_{\Diamond\psi} \in R^+[x_0]$ such that $\min(Rx_0y_{\Diamond\psi}, V(\psi, y_{\Diamond\psi})) > \alpha_{i-1}$. Then let $Y = \{y_{\Box\psi} \in R^+[x_0] : \Box\psi \in \Sigma_\Box\} \cup \{y_{\Diamond\psi} \in R[x_0] : \Diamond\psi \in \Sigma_\Diamond\}$, noting that Y is finite and $|Y| \leq |\Sigma_\Box \cup \Sigma_\Diamond|$.

Now we define $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{T}, \widehat{V}\rangle$ with $\widehat{W} = \bigcup_{y \in Y} \widehat{W}_y \cup \{x_0\}$ and

$$\widehat{R}xz = \begin{cases} Rx_0z, & \text{if } x = x_0 \text{ and } z \in R^+[x_0] \\ \widehat{R}_yxz & \text{if } x, z \in \widehat{W}_y, \text{ for some } y \in Y \\ 0 & \text{otherwise} \end{cases}$$

$$\widehat{T}(x) = \begin{cases} V_{x_0}[\Sigma_\Box \cup \Sigma_\Diamond] \cup \{0, 1\} & \text{if } x = x_0 \\ \widehat{T}_y(x), & \text{if } x \in \widehat{W}_y \text{ for some } y \in Y \end{cases}$$

$$\widehat{V}(p, x) = \begin{cases} V(p, x_0) & \text{if } x = x_0 \\ \widehat{V}_y(p, x) & \text{if } x \in \widehat{W}_y, \text{ for some } y \in Y. \end{cases}$$

Note that, since for all $y \in W$, $\langle \widehat{W}_y, \widehat{R}_y \rangle \subseteq \langle W_y, R_y \rangle \subseteq \langle W, R \rangle$, it follows that $\langle \widehat{W}_y, \widehat{R}_y \rangle \subseteq \langle \widehat{W}, \widehat{R} \rangle \subseteq \langle W, R \rangle$ for all $y \in Y$. Furthermore, because $\widehat{W}_y$ is finite for all $y \in Y \subseteq W$, $\widehat{W}$ is finite. Therefore, it is clear that, $(\star)$, $\widehat{R}^+[x_0] = Y \subseteq R^+[x_0]$ and for all $y \in Y$, $\widehat{R}x_0y = Rx_0y$ and $\widehat{V}(\varphi, y) = V(\varphi, y)$. Then, by an induction on the length of φ, we further show that for all $\varphi \in \Sigma$: $\widehat{V}(\varphi, x_0) = V(\varphi, x_0)$.

The base case follows directly from the definition of $\widehat{V}$. For the inductive step, let $\varphi \in \Sigma$ be of the form $\varphi = \Box\psi$ (the non-modal cases follow directly, using the induction hypothesis). We need to consider two cases. First, let $Rx_0y \to_{\mathsf{G}} V(\psi, y) = 1$ for all $y \in R^+[x_0]$. This implies that for all $y \in R^+[x_0]$: $Rx_0y \leq V(\psi, y)$, and by $(\star)$, that $\widehat{R}x_0y \to_{\mathsf{G}} \widehat{V}(\psi, y) = 1$ for all $y \in \widehat{R}^+[x_0]$ and thus for all $y \in \widehat{W}$. Because $1 \in \widehat{T}(x_0)$, we conclude that

$$\widehat{V}(\Box\psi, x_0) = \max\{r \in \widehat{T}(x_0) : r \leq \inf\{\widehat{R}x_0y \to_{\mathsf{G}} \widehat{V}(\psi, y) : y \in \widehat{W}\}\} = 1.$$

For the second case, let $V(\Box\psi, x_0) = \inf\{Rx_0y \to_{\mathsf{G}} V(\psi, y) : y \in W\} = \alpha_i < 1$ for some $i \in \{1, \ldots, m-1\}$, call it i_0. By $(\star)$, it follows that for all $y \in \widehat{W}$, $\widehat{R}x_0y \to_{\mathsf{G}} \widehat{V}(\psi, y) = Rx_0y \to_{\mathsf{G}} V(\psi, y)$. Because $\widehat{W} \subseteq W$, this implies that $\inf\{\widehat{R}x_0y \to_{\mathsf{G}} \widehat{V}(\psi, y) : y \in \widehat{W}\} \geq \inf\{Rx_0y \to_{\mathsf{G}} V(\psi, y) : y \in W\} = \alpha_{i_0}$. Furthermore, because of our choice of $y_{\Box\psi} \in \widehat{W}$, we know that $\widehat{R}x_0y_{\Box\psi} \to_{\mathsf{G}} \widehat{V}(\psi, y_{\Box\psi}) = Rx_0y_{\Box\psi} \to_{\mathsf{G}} V(\psi, y_{\Box\psi}) < \alpha_{i_0+1}$. Thus $\alpha_{i_0} \leq \inf\{\widehat{R}x_0y \to_{\mathsf{G}} \widehat{V}(\psi, y) : y \in \widehat{W}\} < \alpha_{i_0+1}$ and, by the construction of $\widehat{T}$ ($\widehat{T}(x_0) = \{\alpha_1, \ldots, \alpha_m\}$),

$$\widehat{V}(\Box\psi, x) = \max\{r \in \widehat{T}(x_0) : r \leq \inf\{\widehat{R}x_0y \to_{\mathsf{G}} \widehat{V}(\psi, y) : y \in \widehat{W}\}\} = \alpha_{i_0}.$$

The diamond-case case follows similarly to the box-case and is therefore omitted. Note also that since $\langle \widehat{W}, \widehat{R} \rangle \subseteq \langle W, R \rangle$, crispness is clearly preserved. Finally, we note that $|\widehat{W}| \leq |Y||\Sigma|^n + 1 \leq |\Sigma||\Sigma|^n = |\Sigma|^{hg(\mathfrak{M})}$ and $|\widehat{T}(x_0)| \leq |\Sigma_\Box \cup \Sigma_\Diamond| + 2$, thus $|\widehat{T}(x)| \leq |\Sigma|$ for all $x \in \widehat{W}$. This concludes the induction on the height of the model and therefore the proof of Lemma 3. $\Box$

We now have all the tools required to prove Theorem 1. Suppose first that $\not\models_{\mathsf{GFK}} \varphi$. By Lemma 1(b), φ is not valid in a GFK-tree model of finite height, and hence, by Lemma 2, $\not\models_{\mathsf{GK}} \varphi$. Conversely, suppose that $\not\models_{\mathsf{GK}} \varphi$. By Lemma 1(b), φ is not valid in a GK-tree model $\mathfrak{M}$ with $hg(\mathfrak{M}) \leq \ell(\varphi)$. But then, by Lemma 3, φ is not valid in a GFK-tree model $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{T}, \widehat{V} \rangle$ with (since $|\Sigma(\varphi)| \leq \ell(\varphi)+2$) $|\widehat{W}| \leq (\ell(\varphi)+2)^{\ell(\varphi)}$ and $|\widehat{T}(x)| \leq \ell(\varphi)+2$ for all $x \in \widehat{W}$. This completes the reasoning for (a), and (b) follows in exactly the same manner, using the fact that Lemmas 2 and 3 preserve crispness.

5 A Crisp Gödel S5 Logic

The crisp Gödel modal logic $\mathsf{GS5^C}$ is characterized by validity in $\mathsf{GK^C}$-models where R is an equivalence relation. In fact, it is easily seen that $\mathsf{GS5^C}$-validity corresponds to validity in *universal* $\mathsf{GS5^C}$*-models* where all worlds are related (i.e., GK^{C}-models $\mathfrak{M}$ where $R_{\mathfrak{M}} = W_{\mathfrak{M}} \times W_{\mathfrak{M}}$). Such models may be written $\mathfrak{M} = \langle W, V \rangle$ with simplified valuation clauses

$$V(\Box\varphi, x) = \inf\{V(\varphi, y) : y \in W\} \quad \text{and} \quad V(\Diamond\varphi, x) = \sup\{V(\varphi, y) : y \in W\}.$$

$\mathsf{GS5}^{\mathsf{C}}$ can be axiomatized as an extension of the intuitionistic modal logic MIPC [5, 16] with prelinearity and $\Box(\Box\varphi \vee \psi) \to (\Box\varphi \vee \Box\psi)$ [6]. It may also be viewed as the one-variable fragment of first-order Gödel logic $\mathsf{G}\forall$ (see [11]). Given a formula $\varphi \in \mathrm{Fml}_{\Box\Diamond}$, let φ^* be the first-order formula obtained by replacing each propositional variable p with the predicate $P(x)$, $\Box$ with $\forall x$, and $\Diamond$ with $\exists x$. Then $\models_{\mathsf{GS5}^{\mathsf{C}}} \varphi$ if and only if $\models_{\mathsf{G}\forall} \varphi^*$. Similarly, if φ is a first-order formula with one variable, let φ° be the modal formula obtained by replacing each $P(x)$ with p, $\forall x$ with $\Box$, and $\exists x$ with $\Diamond$. Then $\models_{\mathsf{G}\forall} \varphi$ if and only if $\models_{\mathsf{GS5}^{\mathsf{C}}} \varphi^\circ$.

We define a $\mathsf{GFS5}^{\mathsf{C}}$-*model* as a $\mathsf{GFK}^{\mathsf{C}}$-model $\mathfrak{M} = \langle W, R, T, V \rangle$ such that $\langle W, R, V \rangle$ is a $\mathsf{GS5}^{\mathsf{C}}$-model, and also $T(x) = T(y)$ whenever Rxy. Again, $\mathsf{GFS5}^{\mathsf{C}}$-validity amounts to validity in *universal* $\mathsf{GFS5}^{\mathsf{C}}$-*models*, written $\mathfrak{M} = \langle W, T, V \rangle$, where T may now be understood as a single fixed finite subset of $[0, 1]$, and

$$\begin{aligned} V(\Box\varphi, x) &= \max\{r \in T : r \leq \inf\{V(\varphi, y) : y \in W\}\} \\ V(\Diamond\varphi, x) &= \min\{r \in T : r \geq \sup\{V(\varphi, y) : y \in W\}\}. \end{aligned}$$

Note in particular that in both $\mathsf{GS5}^{\mathsf{C}}$-models and $\mathsf{GFS5}^{\mathsf{C}}$-models, the truth values of box-formulas and diamond-formulas are independent of the world.

Lemma 4. *For any universal* $\mathsf{GFS5}^{\mathsf{C}}$*-model* $\mathfrak{M}$*, there is a universal* $\mathsf{GS5}^{\mathsf{C}}$*-model* $\widehat{\mathfrak{M}}$ *with* $W_{\mathfrak{M}} \subseteq W_{\widehat{\mathfrak{M}}}$*, such that* $V_{\widehat{\mathfrak{M}}}(\varphi, x) = V_{\mathfrak{M}}(\varphi, x)$ *for all* $\varphi \in \mathrm{Fml}_{\Box\Diamond}$ *and* $x \in W_{\mathfrak{M}}$*.*

Proof. We proceed similarly to the proof of Lemma 2, but since there is no accessibility relation here, an induction is not required. Given a universal $\mathsf{GFS5}^{\mathsf{C}}$-model $\mathfrak{M}$, we construct the universal $\mathsf{GS5}^{\mathsf{C}}$-model $\widehat{\mathfrak{M}}$ directly by taking infinitely many copies of $\mathfrak{M}$. Consider $T_{\mathfrak{M}} = \{\alpha_1, \ldots, \alpha_n\}$ with $0 = \alpha_1 < \ldots < \alpha_n = 1$ and, using Lemma 1(c), define a family of order-embeddings $\{h_k\}_{k \in \mathbb{Z}^+}$ exactly as in the proof of Lemma 2. For all $k \in \mathbb{Z}^+$, we define a universal $\mathsf{GS5}^{\mathsf{C}}$-model $\widehat{\mathfrak{M}}_k = \langle \widehat{W}_k, \widehat{V}_k \rangle$ such that each $\widehat{W}_k$ is a copy of $W_{\mathfrak{M}}$ with distinct worlds and $\widehat{V}_k(\varphi, x_k) = h_k(V_{\mathfrak{M}}(\varphi, x))$ for each copy x_k of $x \in W_{\mathfrak{M}}$ and $\varphi \in \mathrm{Fml}_{\Box\Diamond}$. We also let $\widehat{W}_0 = W_{\mathfrak{M}}$ and $\widehat{V}_0 = V_{\mathfrak{M}}$. Then $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{V} \rangle$ where

$$\widehat{W} = \bigcup_{k \in \mathbb{N}} \widehat{W}_k \qquad \text{and} \qquad \widehat{V}(p, x) = \widehat{V}_k(p, x) \;\text{ for } x \in \widehat{W}_k.$$

It then suffices to prove that $\widehat{V}(\varphi, x) = V(\varphi, x)$ for all $\varphi \in \mathrm{Fml}_{\Box\Diamond}$ and $x \in W$, proceeding by an induction on $\ell(\varphi)$ similar to the proof of Lemma 2. $\Box$

Lemma 5. *Let* $\Sigma \subseteq \mathrm{Fml}_{\Box\Diamond}$ *be a finite fragment. Then, for any universal* $\mathsf{GS5}^{\mathsf{C}}$*-model* $\mathfrak{M}$*, there is a finite universal* $\mathsf{GFS5}^{\mathsf{C}}$*-model* $\widehat{\mathfrak{M}}$ *with* $W_{\widehat{\mathfrak{M}}} \subseteq W_{\mathfrak{M}}$*, such that* $V_{\widehat{\mathfrak{M}}}(\varphi, x) = V_{\mathfrak{M}}(\varphi, x)$ *for all* $\varphi \in \Sigma$ *and* $x \in W_{\widehat{\mathfrak{M}}}$*. Moreover,* $|W_{\widehat{\mathfrak{M}}}| + |T_{\widehat{\mathfrak{M}}}| \leq 2|\Sigma|$*.*

Proof. Let $\Sigma \subseteq \text{Fml}_{\Box\Diamond}$ be a finite fragment, $\mathfrak{M} = \langle W, V \rangle$ a universal $\mathsf{GS5^C}$-model, and fix $x_0 \in W$. First, define $\Sigma_\Box$, $\Sigma_\Diamond$, Σ_{Var}, and $V_x[\Delta]$ as in Lemma 3 and let $V_{x_0}[\Sigma_\Box \cup \Sigma_\Diamond] \cup \{0,1\} = \{\alpha_1, \ldots, \alpha_n\}$ with $0 = \alpha_1 < \ldots < \alpha_n = 1$. As in Lemma 3, we choose a finite number of $y \in W$. For each $\Box\psi \in \Sigma_\Box$ such that $V(\Box\psi, x_0) = \alpha_i < 1$, choose a $y = y_{\Box\psi} \in W$ such that $V(\psi, y_{\Box\psi}) < \alpha_{i+1}$, and for each $\Diamond\psi \in \Sigma_\Diamond$, such that $V(\Diamond\psi, x_0) = \alpha_i > 0$, choose a $y = y_{\Diamond\psi} \in W$ such that $V(\psi, y_{\Diamond\psi}) > \alpha_{i-1}$. Then let $\widehat{W} = \{x_0\} \cup \{y_{\Box\psi} \in W : \Box\psi \in \Sigma_\Box\} \cup \{y_{\Diamond\psi} \in W : \Diamond\psi \in \Sigma_\Diamond\}$. Clearly $\widehat{W} \subseteq W$ is finite. We define $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{T}, \widehat{V} \rangle$ where $\widehat{T} = V_{x_0}[\Sigma_\Box \cup \Sigma_\Diamond] \cup \{0,1\}$, and $\widehat{V}$ is V restricted to $\widehat{W}$. It then follows by induction on $\ell(\varphi)$ (omitted here) as in Lemma 3 that $\widehat{V}(\varphi, x) = V(\varphi, x)$ for all $x \in \widehat{W}$ and $\varphi \in \Sigma$. Moreover, $|\widehat{W}| \leq |\Sigma_\Box \cup \Sigma_\Diamond| + 1 \leq |\Sigma|$ and $|\widehat{T}| \leq |\Sigma_\Box \cup \Sigma_\Diamond| + 2 \leq |\Sigma|$, and therefore $|\widehat{W}| + |\widehat{T}| \leq 2|\Sigma|$. ◻

Hence, immediately by Lemmas 4 and 5:

Theorem 3. *For each $\varphi \in \text{Fml}_{\Box\Diamond}$: $\models_{\mathsf{GS5^C}} \varphi$ iff $\models_{\mathsf{GFS5^C}} \varphi$ iff φ is valid in all universal $\mathsf{GFS5^C}$-models $\mathfrak{M}$ where $|W_\mathfrak{M}| + |T_\mathfrak{M}| \leq 2(\ell(\varphi) + 2)$.*

Finally, to check non-deterministically if a formula $\varphi \in \text{Fml}_{\Box\Diamond}$ is not $\mathsf{GS5^C}$-valid, it suffices, using Lemmas 4 and 5, to guess a universal $\mathsf{GFS5^C}$-model $\mathfrak{M}$ with $|W_\mathfrak{M}| = \ell(\varphi)$ and values in $U = \{0, \frac{1}{\ell(\varphi)^2+1}, \ldots, \frac{\ell(\varphi)^2}{\ell(\varphi)^2+1}, 1\}$ and then to guess a world $x \in W_\mathfrak{M}$ and check whether $V_\mathfrak{M}(\varphi, x) < 1$. This means choosing $\ell(\varphi)|\text{Var}(\varphi)|$ values in U for $V_\mathfrak{M}$ and also a subset of U for $T_\mathfrak{M}$. Choosing these values and the world $x \in W_\mathfrak{M}$ and then computing the value of $V_\mathfrak{M}(\varphi, x) = 1$ can be achieved in time polynomial in $\ell(\varphi)^2$. So validity in $\mathsf{GS5^C}$ is in co-NP. Since checking validity in Gödel logic is co-NP hard (see, e.g., [11]), we obtain:

Theorem 4. *Validity in $\mathsf{GS5^C}$ and the one-variable fragment of first-order Gödel logic is decidable and indeed co-NP-complete.*

6 Concluding Remarks

In this paper, we have established the decidability of validity in the Gödel modal logics GK and $\mathsf{GK^C}$ based, respectively, on fuzzy and crisp Kripke frames. We have also established decidability and co-NP completeness for validity in the one-variable fragment of first-order Gödel logic and, equivalently, the logic $\mathsf{GS5^C}$ based on crisp Kripke frames where accessibility is an equivalence relation. In ongoing work, we aim to determine the complexity of validity in GK and $\mathsf{GK^C}$ (both of which we conjecture to be PSPACE complete), possibly via Gentzen-style proof systems. We also intend to extend our approach to other logics. From a modal perspective, we plan to consider logics with multiple modalities (in particular, Gödel description logics) and modalities whose accessibility relations satisfy conditions such as reflexivity, symmetry, and transitivity. Moreover, in the propositional setting, we intend to treat modal logics based on so-called "projective logics" (see [1]) where similar methods should apply.

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